A comonadic view of simulation and quantum resources

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We study simulation and quantum resources in the setting of the sheaf-theoretic approach to contextuality and non-locality. Resources are viewed behaviourally, as empirical models. In earlier work a notion of a morphism for these empirical models was proposed and studied. We generalize and simplify the earlier approach, by starting with a very simple notion of morphism, and then extending it to a more useful one by passing to a co-Kleisli category with respect to a comonad of measurement protocols. We show that these morphisms capture notions of simulation between empirical models obtained via "free" operations in a resource theory of contextuality, including the type of classical control used in measurement-based quantum computation schemes.

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A key objective in the field of quantum information and computation is to understand the advantage which can be gained in information processing tasks by the use of quantum resources. While a range of examples have been studied, to date a systematic understanding of quantum advantage is lacking.

One approach to achieving such a general understanding is through *resource theories* [11, 2], in which one considers a set of operations by which one system can be transformed into another. In particular, one considers "free operations", which can be performed without consuming any additional resources of the kind in question. If resource B can be constructed from A using only free operations, then we say that A is convertible to B, or B is reducible to A. This point of view is studied in some generality in [8, 10].

Another natural approach, which is familiar in computation theory, is to consider a notion of *simulation*; one asks if the behaviour of B can be produced by some protocol using A as a resource.

Both these points of view can be considered in relation to quantum advantage. Our focus in this paper is on quantum resources which take the form of *non-local*, or more generally *contextual*, correlations. Contextuality is one of the key signatures of non-classicality in quantum mechanics [14, 6], and has been shown to be a necessary ingredient for quantum advantage in a range of information processing tasks [15, 12, 7, 2]. In previous work [2], we showed how this advantage could be quantified in terms of the *contextual fraction*, and we also introduced a range of free operations, which were shown to have the required property of being non-increasing with respect to the contextual fraction. Thus this work provided some of the basic ingredients for a resource theory of quantum advantage, with contextuality as the resource.

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In [13], one of the present authors introduced a notion of simulation between (possibly contextual) behaviours, as morphisms between empirical models, in the setting of the "sheaf-theoretic" approach to contextuality introduced in [3]. This established a basis for a simulation-based approach to comparing resources.

In this paper, we bring these two approaches together.

- On the simulation side, we enhance the treatment given in [13] by introducing a *measurement protocols* construction on empirical models. Measurement protocols were first introduced in a different setting in [4]. This construction captures the intuitive notion, widely used in an informal fashion in concrete results in quantum information (e.g. [5]), of using a "box" or device by performing some measurement on it, and then, depending on the outcome, choosing some further measurements to perform. This form of adaptive behaviour also plays a crucial role in measurement-based quantum computing [16].

  We show that this construction yields a comonad on the category of empirical models, and hence we are able to describe a very general notion of simulation of B by A in terms of co-Kleisli maps from A to B.

- We consider the algebraic operations previously introduced in [2] and introduce a new operation allowing a conditional measurement, a one-step version of adaptivity. We present an equational theory for these operations and use this to obtain normal forms for resource expressions.

- Using these normal forms, we obtain one of our main results: we show that the algebraic notion of convertibility coincides with the existence of a simulation morphism.

- We also prove some further results, including a form of no-cloning theorem at the abstract level of simulations.

Mathematical setting

We now proceed to elaborate briefly on these notions.

The types of behaviours are given by *scenarios* (X,Σ,O), where X is a set of measurement labels, Σ is a simplicial complex with vertices X giving the compatible sets of measurements, and $O = \{O_x\}_{x\in X}$ is the set of possible outcomes for each measurement $x \in X$. The set of joint outcomes for a context $C \in \Sigma$ is $\prod_{x\in C} O_x$.

A behaviour over a scenario (X,Σ,O) is given by an *empirical model* $e = \{e_C\}_{C\in\Sigma}$, where e_C is a probability distribution on $\prod_{x\in C} O_x$. This must satisfy the compatibility (or local consistency) condition: for all $C,C' \in \Sigma$, $e_C|_{C\cap C'} = e_{C'}|_{C\cap C'}$, where restriction is given by marginalization. Physically, this corresponds to the no-disturbance condition, and in the case of Bell scenarios (interpreted as spatially separated), to no-signalling. An empirical model e is *contextual* if there is no probability distribution d on $\prod_{x\in X} O_x$, the set of global assignments of outcomes to all measurements, which marginalizes to the empirical behaviour, i.e. such that $d|_C = e_C$ for all $C \in \Sigma$. Contextuality thus arises when there is local consistency but global inconsistency.

A morphism of scenarios $(\pi,h) : (X,\Sigma,O) \to (Y,\Delta,P)$ is given by a simplicial map $\pi : \Delta \to \Sigma$, and for each $y \in Y$ a map $h_y : O_{\pi(y)} \to P_y$. Such a morphism induces a natural action on empirical models: if e is an empirical model on (X,Σ,O), then $(\pi,h)^*e$ is an empirical model on (Y,Δ,P), given by $(\pi,h)^*(e)_C = \mathsf{D}(\gamma)(e_{\pi(C)})$, the push-forward of the probability measure $e_{\pi(C)}$ along the map $\gamma : \prod_{x\in\pi(C)} O_x \to \prod_{y\in C} P_y$ given by $\gamma(s)_y = h_y\left(s_{\pi(y)}\right)$ This gives a category **Emp**, with objects $e : (X,\Sigma,O)$, and morphisms $(\pi,h) : e \to e'$ such that $(\pi,h)^*(e) = e'$.

To obtain our full notion of simulation, we define a comonad MP on **Emp**, which builds the set of *measurement protocols* over a given scenario. These protocols proceed recursively by first performing a measurement over the given scenario, and then conditioning their further measurements on the outcome. Thus they incorporate adaptive classical processing, of the kind used e.g. in Measurement-Based Quantum Computing. We omit the details of the MP construction in this abstract; see [1].

Given empirical models e and d, a *simulation* of e by d is a map $d \otimes c \to e$ in $\mathbf{Emp}_{\mathsf{MP}}$, the coKleisli category of MP, i.e. a map $\mathsf{MP}(d \otimes c) \to e$ in **Emp**, for some noncontextual model c. We denote the existence of a simulation of e by d as $d \rightsquigarrow e$, read "d simulates e". The use of the noncontextual model c is to allow for classical randomness in the simulation.

Further directions

We briefly indicate some directions of ongoing further work.

Degrees of Contextuality

The relation $d \rightsquigarrow e$ is a preorder on empirical models. The induced equivalence classes are the *degrees of contextuality*. They are partially ordered by the existence of simulations between representatives. This partial order can be seen as a fundamental structure in the study of quantum resources. We may ask how rich this order is. Existing results in the non-locality literature such as [9] can be leveraged to prove the following theorem.

Theorem 1 *The order contains both infinite strict chains, and infinite antichains.*

This is just a preliminary observation. Many questions arise, and there are natural variations and refinements.

Generalized Vorob'ev theory

The property of (non)contextuality itself can be equivalently formulated as the existence of a simulation by an empirical model over the empty scenario [1]. This suggests that much of contextuality theory can be generalized to a "relativized" form, i.e. essentially working in slice categories.

As an example, consider the classic theorem of Vorob'ev [17]. It characterizes those scenarios over which all empirical models are noncontextual, in terms of an acyclicity condition on the underlying simplicial complex. This can be formulated as characterizing those scenarios such that every model over them can be simulated by a model over the empty scenario.

More generally, we can ask for conditions on scenarios (X, Σ, O) and (Y, Δ, P) such that every empirical model over (Y, Δ, P) can be simulated by some empirical model over (X, Σ, O). We have some preliminary results on this question.

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